Unrelated Machine Scheduling of Jobs with Uniform Smith Ratios

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joint work with Christos Kalaitzis and Ola Svensson

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FÉDÉRALE DE LAUSANNE

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Scheduling on unrelated machines

- n jobs
- m machines
- allocate each job to one machine
- minimize certain objective

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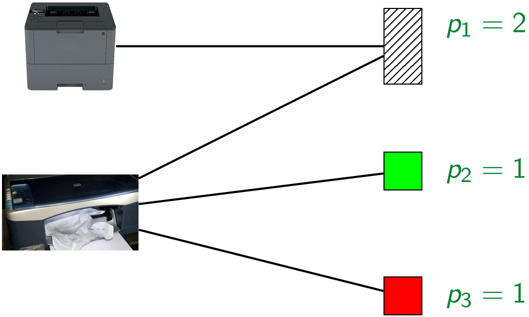
Objective: understand the approximability of these problems

Scheduling on unrelated machines



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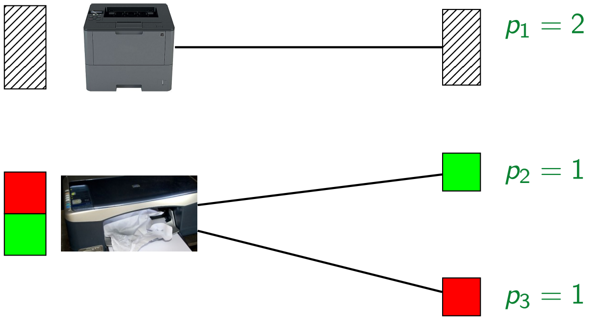

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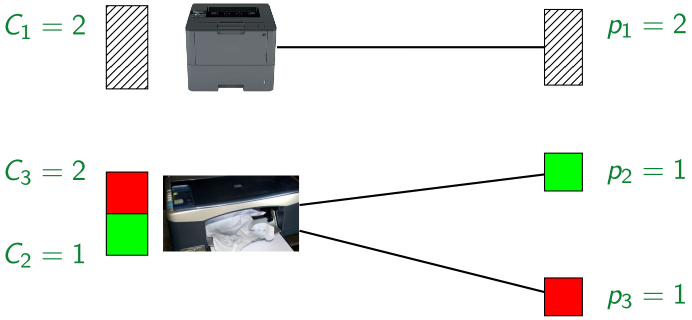
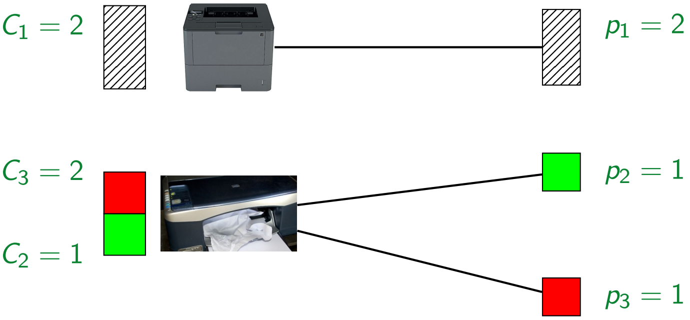


C_j: completion time of job j



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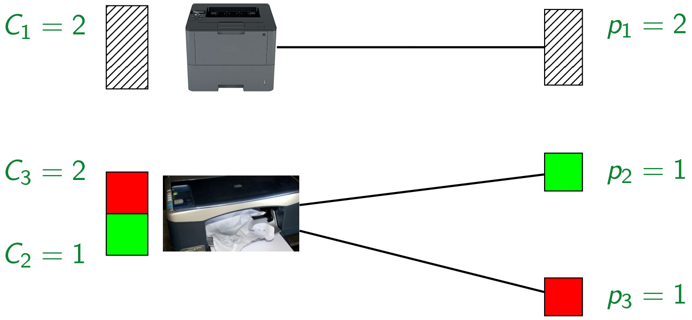
Minimize:

- **makespan**: $\max_j C_j$



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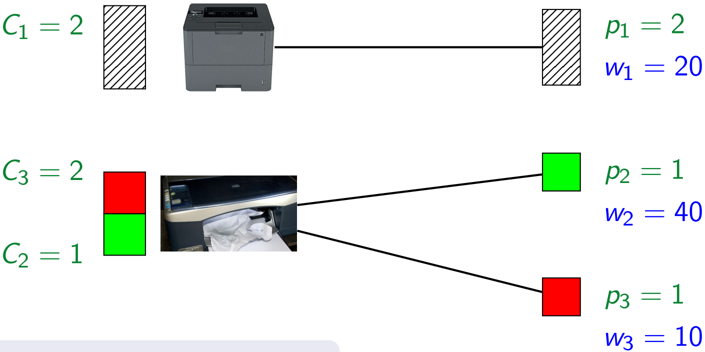
Scheduling on unrelated machines



C_j: completion time of job j

Minimize:

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- **weighted sum of completion times**: $\sum_j w_j C_j$
  - given weights w_j: importance of job j

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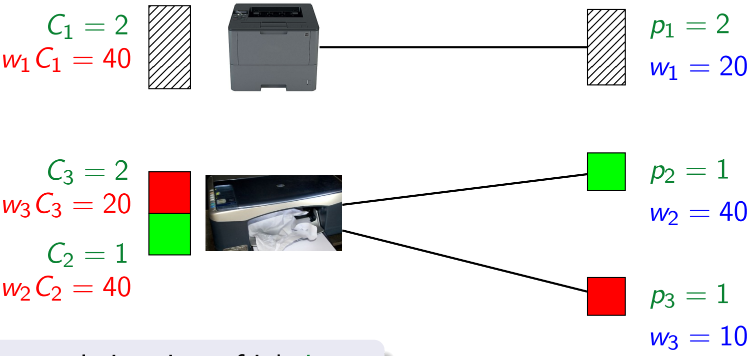

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Minimize:

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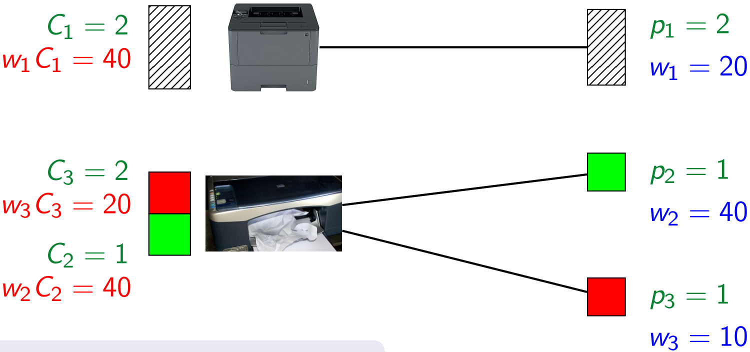

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Single machine

To minimize weighted sum of completion times $\sum_j w_j C_j$:

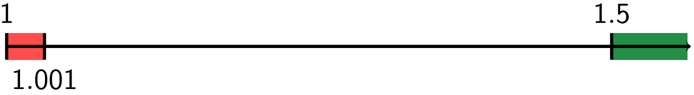
Smith's rule [1956]

Order jobs by w_j / p_j (Smith ratio)

So:

- allocate jobs to machines: **hard part**
- order jobs on every machine: **easy**

State of the art



Hoogeveen et al. 2001

Minimizing $\sum_j w_j C_j$ is hard to approximate within 1.001.

Skutella 2001 / Sethuraman, Squillante 1999

There is a 1.5-approximation algorithm
(*independent* randomized rounding of convex relaxation).

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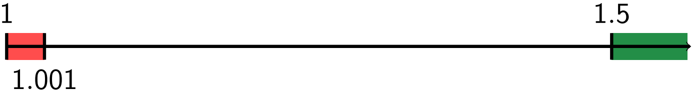

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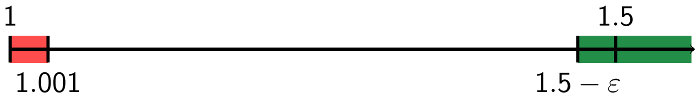

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Our special case: uniform-Smith-ratios

For each machine i, Smith ratios are uniform: $p_{ij} \in \{\alpha_i w_{ij}, \infty\}$.

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- natural: every unit of work has same weight
- jobs: time-consuming $\iff$ important

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So:

- allocate jobs to machines: **hard part**
- order jobs on every machine: ~~**easy**~~ **irrelevant**

Hardness

Other special cases can be easy:

- all $w_j = 1$: in P
- identical parallel machines: has PTAS

but uniform-Smith-ratios inherits hardness of general version:

- still APX-hard
- still *independent* randomized rounding can only yield 1.5
- still the previous relaxations have integrality gap 1.5

Our result

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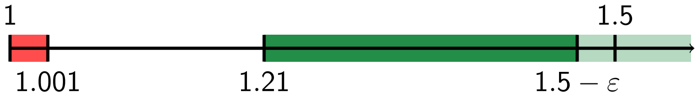
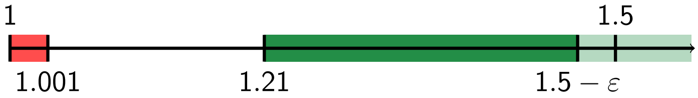


Our main result

There is a $\frac{1+\sqrt{2}}{2} \approx 1.21$-approximation algorithm for unrelated machine scheduling with uniform Smith ratios.

Analysis is **tight**.

Our result



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Bonus

Simultaneous 2-approximation for makespan and 1.21-approximation for $\sum_j w_j C_j$.

Plan of talk:

- Configuration-LP
  - assigns *configurations* (subsets of jobs) to machines
- Shmoys-Tardos rounding
  - randomized rounding of LP solution
- flavor of analysis
  - fix single machine
  - compare two probability distributions on configurations: from LP solution and from our rounding
  - bound ratio of their expected costs

Configuration-LP

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Very strong LP relaxation which assigns whole *configurations* (subsets of jobs) to machines.

- variable $y_{iC} \geq 0$ for each machine i and configuration C
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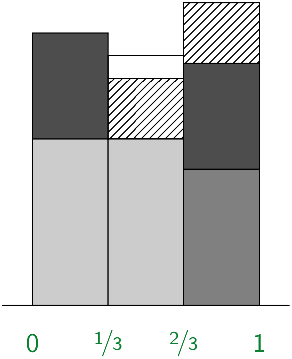
$1.08 \leq$ integrality gap ≤ 1.21 (this work)

Configuration-LP outputs a distribution



Distribution on configurations for a fixed machine $i^\star$

- rectangle: job
- height of rectangle: processing time
- stack of rectangles: configuration
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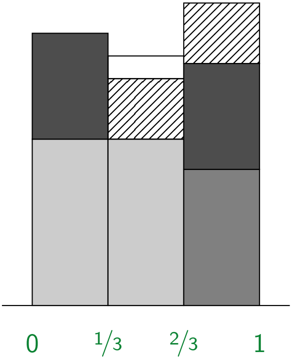

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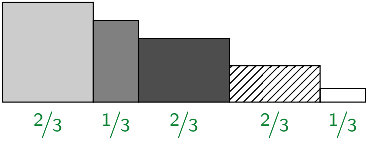

$$x_{ij} = \sum_{C \ni j} y_{iC}$$

Shmoys-Tardos rounding

Shmoys-Tardos rounding

- map the marginals x to
  a fractional matching in a bipartite graph

- randomly round this fractional matching to
  an integral matching (which corresponds to a schedule)

- originally used for 2-approximation for the makespan objective
  (applied to the so-called Assignment-LP)

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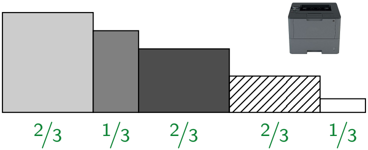

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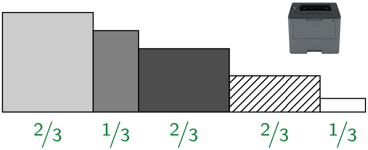

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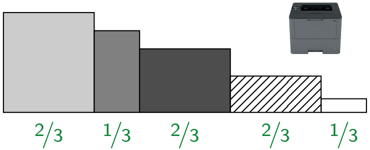

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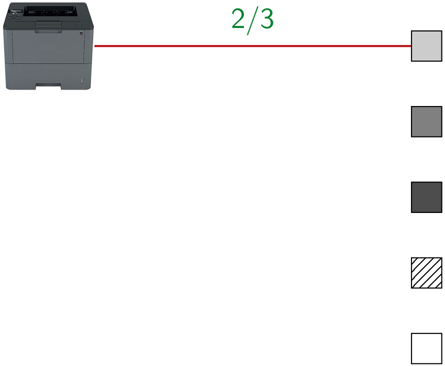

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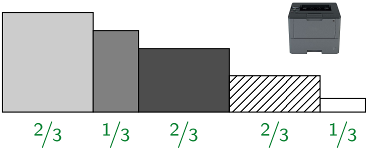

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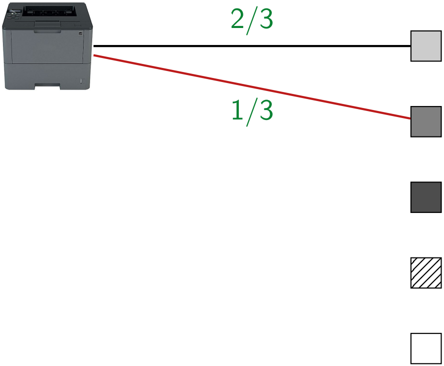

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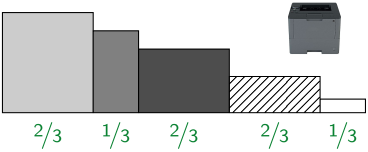

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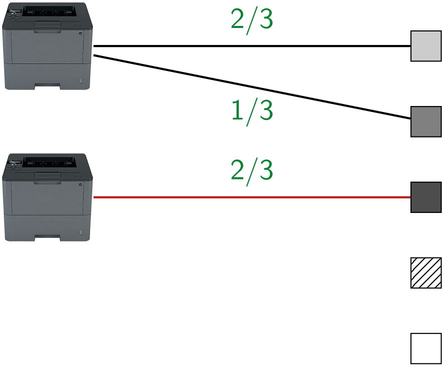

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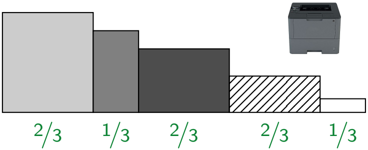

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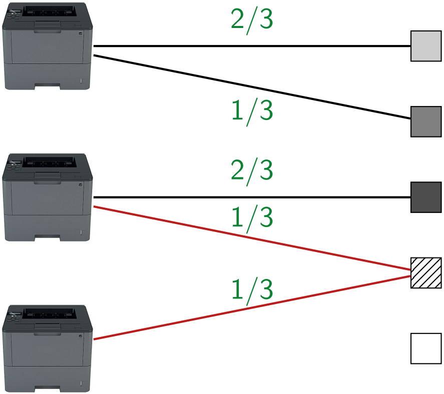

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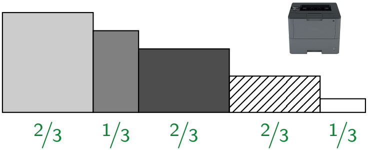

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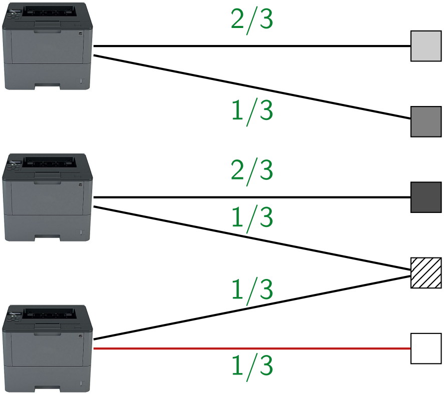

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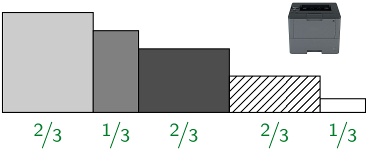

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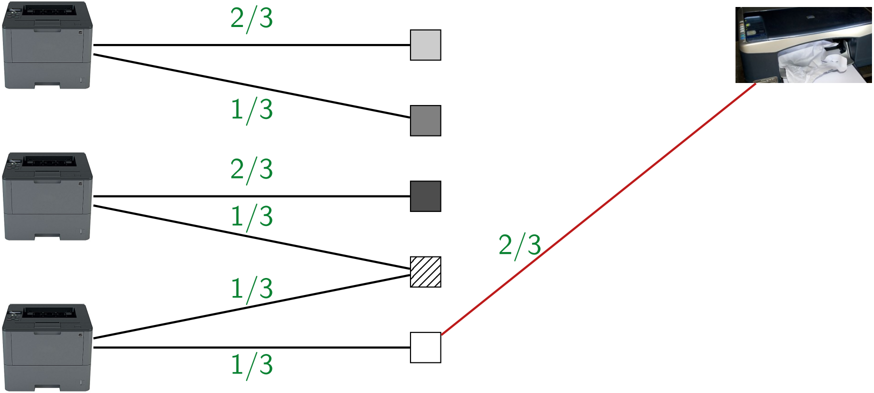

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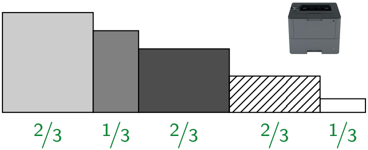

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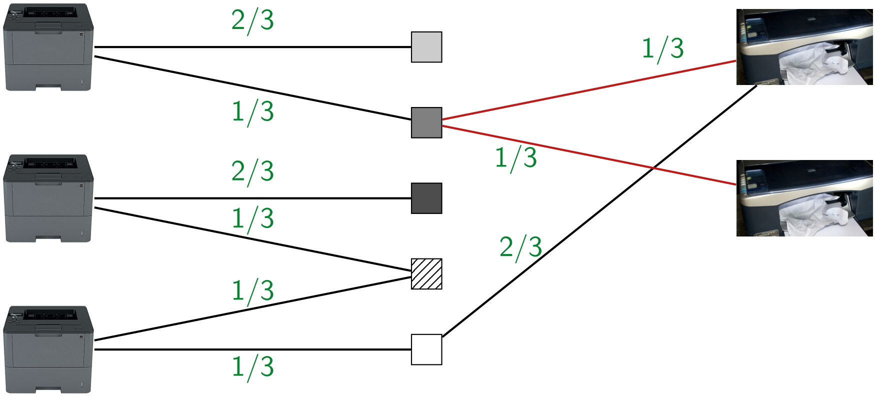

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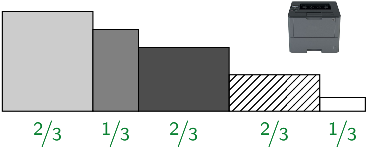

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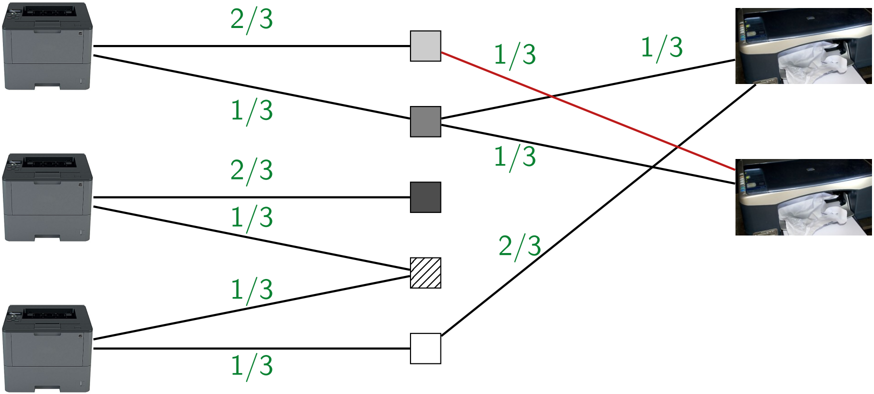

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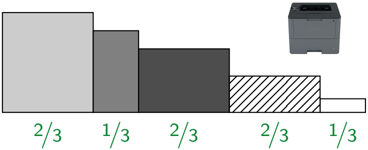

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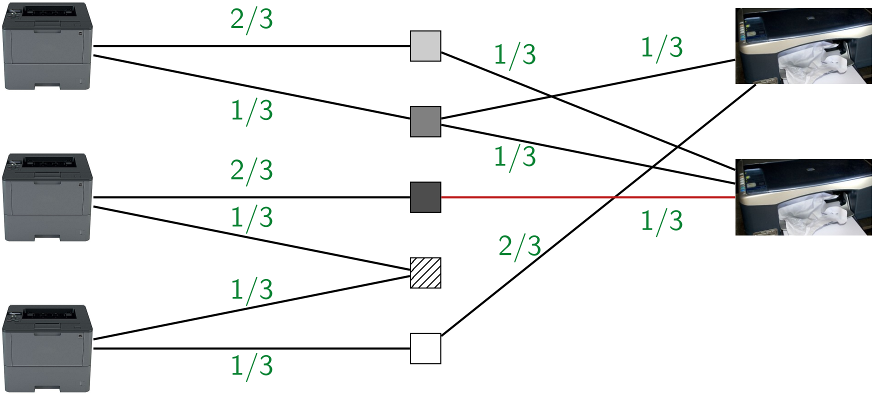

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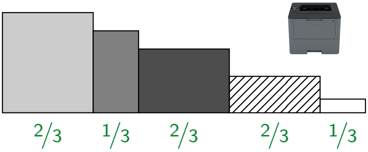

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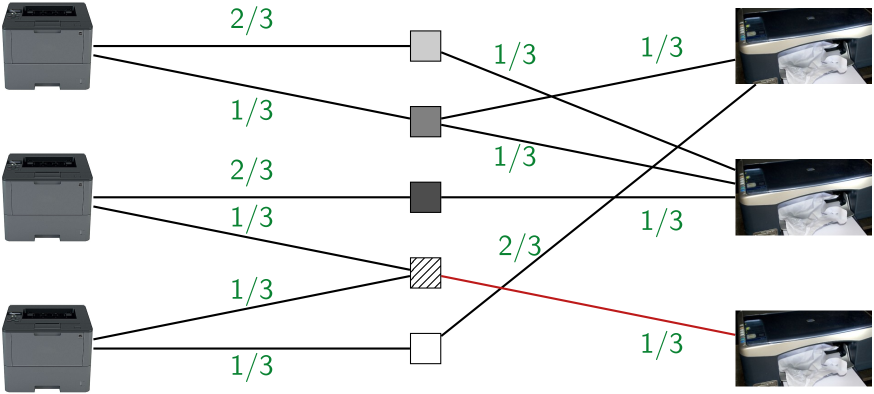

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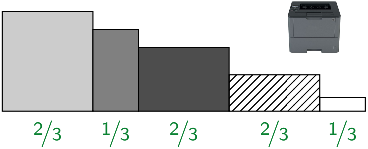

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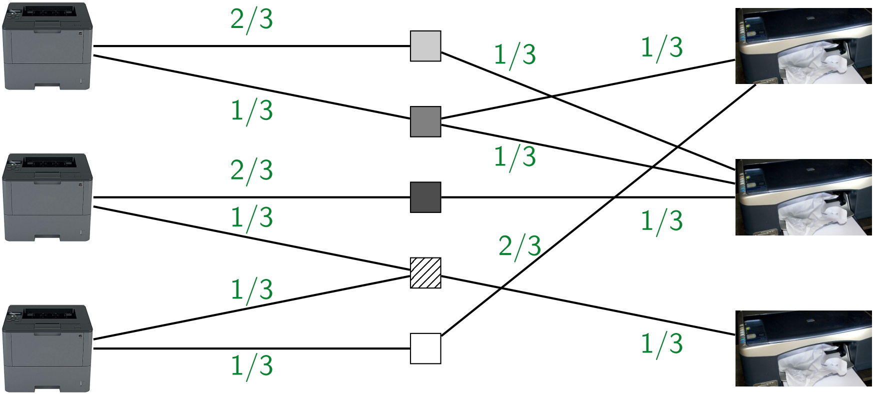

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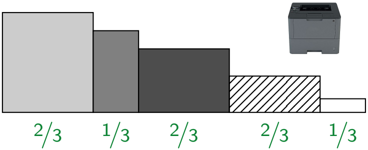

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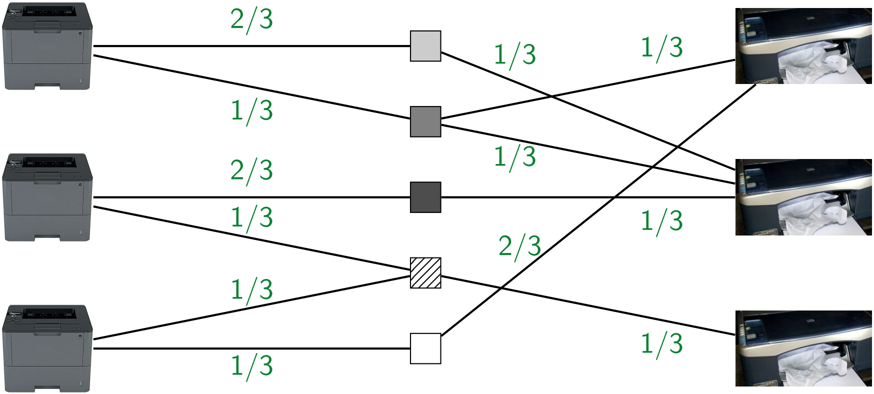

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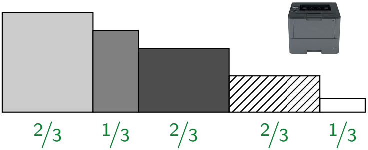

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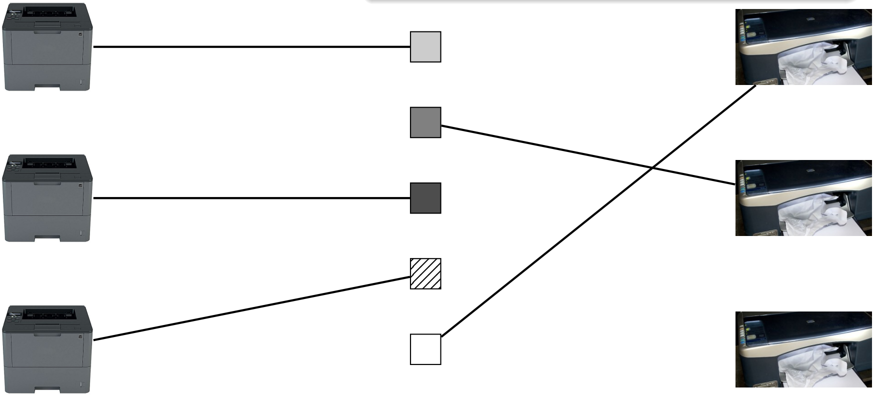

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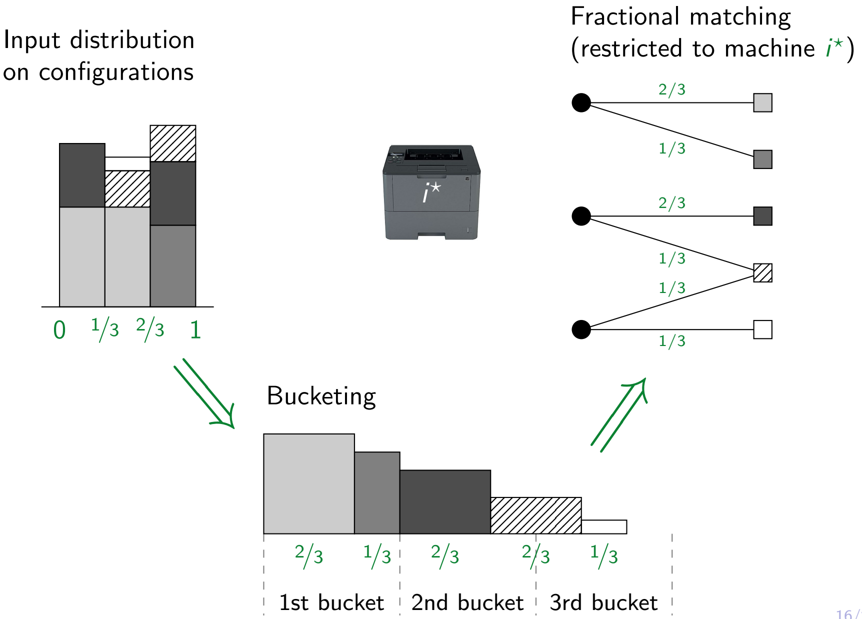

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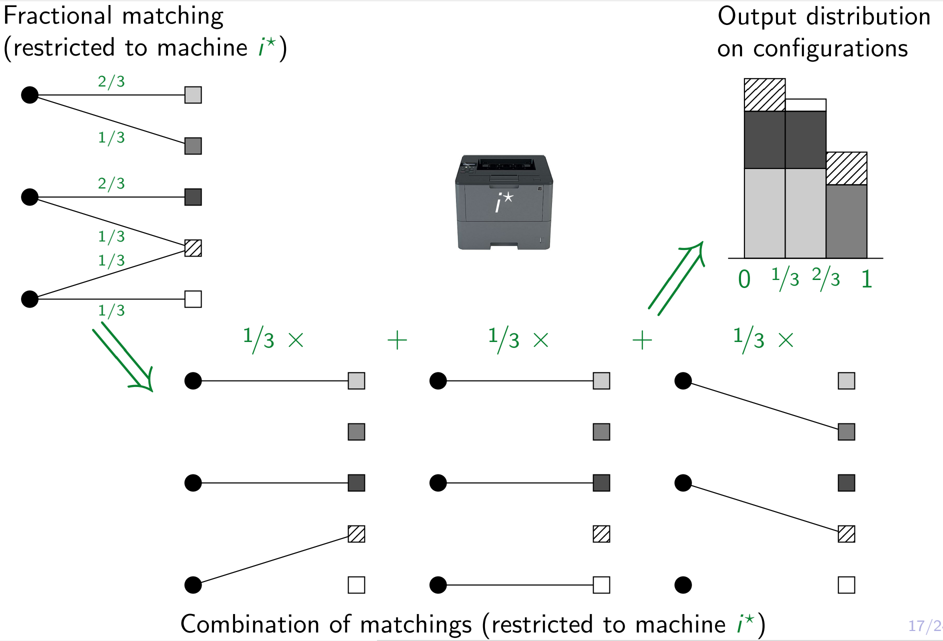
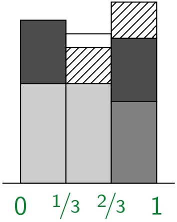


Our analysis

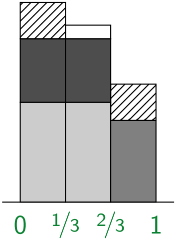
Two configurations

Input distribution
on configurations (y^{in})



$\text{cost}(y^{\text{in}})$: **LP lower bound**

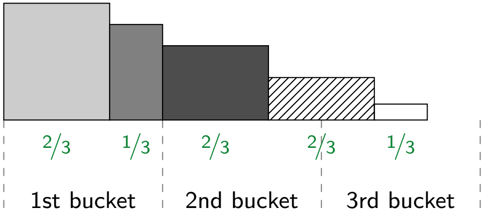
Output distribution
on configurations (y^{out})



$\text{cost}(y^{\text{out}})$: **cost of our solution**

- we want to bound $\frac{\text{cost}(y^{\text{out}})}{\text{cost}(y^{\text{in}})} \leq \frac{1+\sqrt{2}}{2} \approx 1.21$
- y^{in} and y^{out} have same marginals
- y^{out} has a nice bucket structure from our rounding

Bucket structure



Each configuration gets:

- one job from 1st bucket
- one job from 2nd bucket
- one job from 3rd bucket (or none)

k-th largest job of any configuration
$$\geq$$
$(k+1)$-th largest job of any configuration

Nonexistent bad example

Input distribution on configurations

Output distribution on configurations

- left: best possible distribution with marginals $1/2$ on both jobs
- right: worst possible such distribution (would give ratio 1.5)
- good if small variance

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Input distribution
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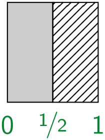
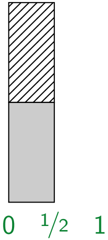


Output distribution
on configurations (impossible)



- left: best possible distribution with marginals $1/2$ on both jobs
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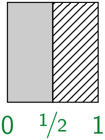

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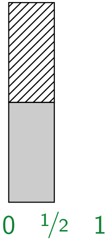

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Our analysis

- we transform both y^{in} and y^{out} while making the ratio worse

$$(y^{\text{in}}, y^{\text{out}}) \to (y_1^{\text{in}}, y_1^{\text{out}}) \to (y_2^{\text{in}}, y_2^{\text{out}}) \to \dots$$

$$\frac{\text{cost}(y^{\text{out}})}{\text{cost}(y^{\text{in}})} \le \frac{\text{cost}(y_1^{\text{out}})}{\text{cost}(y_1^{\text{in}})} \le \frac{\text{cost}(y_2^{\text{out}})}{\text{cost}(y_2^{\text{in}})} \le \dots$$

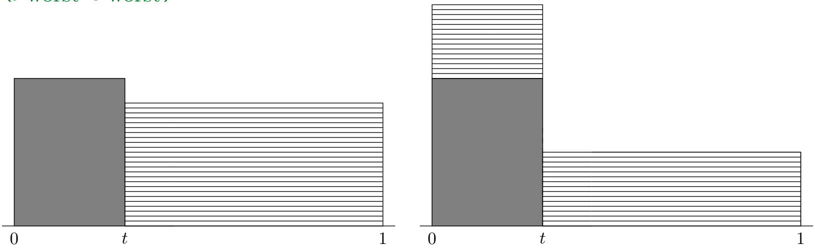
  (main technical part, uses uniform Smith ratios)
- we arrive at a "worst-case" pair for which we can bound the ratio by $\frac{1+\sqrt{2}}{2}$

$$\dots \to (y_{\text{worst}}^{\text{in}}, y_{\text{worst}}^{\text{out}})$$

$$\dots \le \frac{\text{cost}(y_{\text{worst}}^{\text{out}})}{\text{cost}(y_{\text{worst}}^{\text{in}})} \le \frac{1+\sqrt{2}}{2}$$

Our analysis

$(y^{\text{in}}_{\text{worst}}, y^{\text{out}}_{\text{worst}})$ looks like this:



- gray part: single large job
- striped parts: many jobs with *infinitesimal size* $\varepsilon \to 0$

$$\frac{\text{cost}(y^{\text{out}}_{\text{worst}})}{\text{cost}(y^{\text{in}}_{\text{worst}})} \leq \sup_{t\in[0,1),\gamma\geq 0,\lambda\geq 0} \frac{t\gamma^2 + t\gamma\lambda + \frac{\lambda^2}{2}}{t\gamma^2 + \frac{\lambda^2}{2(1-t)}} \leq \frac{1+\sqrt{2}}{2}$$

- analysis **tight**: this corresponds to a scheduling instance

Summary

Our main result

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- $+$ much simpler algorithm and analysis

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Thank you!