Fair Submodular Maximization over Matroid constraint

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Matroid Submodular Maximization

Set function

$f : 2^V \to \mathbb{R}_+$

Monotone

$f(X \cup \{e\}) \geq f(X)$

for any set X and element e

Submodular: Diminishing Returns

$$f(X \cup \{e\}) - f(X) \geq f(Y \cup \{e\}) - f(Y)$$

for any sets X, Y and element e

Maximize f over matroid $\mathcal{I} \subseteq 2^V$

$$\max_{S \in \mathcal{I}} f(S)$$

Examples: uniform matroid |S| ≤ k, partition matroid $|S \cap V_c| \leq u_c$

Influence Maximization

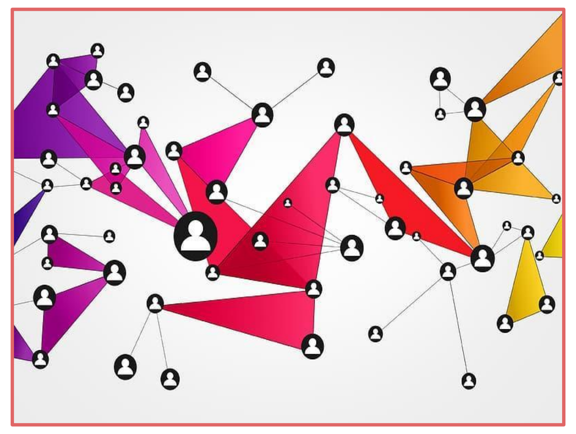
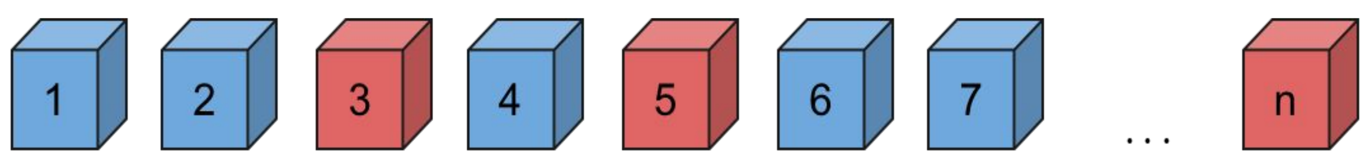


Image Summarization

Fairness Setting

- Each element has a color encoding a sensitive attribute.
- We are given lower and upper bound constraints for each color c:

$$\ell_c \leq |S \cap V_c| \leq u_c \text{ for all } c = 1, \cdots, C$$



Find a solution such that

1. Number of blue elements in range [1, 2]
2. Number of red elements is in range [0, 3]
3. The solution belongs to a matroid

The bounds are not constants

Family of feasible sets: $\mathcal{F}$ = fair sets in matroid $\mathcal{I}$

Prior Work

Cardinality constraint:

- Offline setting: tight (1-1/e)-approx for monotone *f* [Celis et al, 2018].
- Streaming setting: 0.3178-approx for monotone *f*, 0.1921q-approx for non-monotone, where $q = 1 - \max_c \frac{\ell_c}{|V_c|}$ [El Halabi et al, 2020].

General matroid:

- Streaming setting: 0.085-approx for monotone *f* with factor-2 violation of fairness lower bounds [El Halabi et al, 2023].
- **No results** for offline setting or non-monotone objectives.

Our Results

Function	Matroid	Approx. Ratio	Fairness Approx.
Monotone	General	$1 - 1/e$ (Thm. 3.3)	$(1, 1)$ in expectation
Non-monotone	General	$(1 - r - \epsilon)/4$ (Thm. 3.3)	$(1 - \sqrt{3\ln(2C)/\ell_c}, 1 + \sqrt{3\ln(2C)/u_c})$
Monotone	General	$1/(4 + \epsilon)$ (El Halabi et al., 2023)	$(1/2, 1)$
Non-monotone	General	$(1 - \beta)/(8 + \epsilon)$ (Thm. 3.1)	$(\beta, 1)$
Monotone	Uniform	$1 - 1/e$ (Celis et al., 2018a)	$(1, 1)$
Non-monotone	Uniform	$0.401(1 - r)$ (Thm. 3.6)	$(1, 1)$
Monotone decomposable	General	$1 - 1/e$ (Thm. 4.4)	$(1, 1)$
Non-monotone decomposable	General	$(1 - r - \epsilon)/4$ (Thm. 4.4)	$(1, 1)$

$r = \min_{x \in P_{\mathcal{F}} \cup (\mathbb{1} - P_{\mathcal{F}})} \|x\|_\infty$ $P_{\mathcal{F}}$ = convex hull of feasible sets $\beta \in [0, 1/2]$

Lower bounds:

- No approximation better than (1 - r) in sub-exponential time for non-monotone
- No O(1/√n) -approximation using **relax-and-round** approaches

Relax-and-round:

1. Relax: maximize multilinear extension *F* over $P_{\mathcal{F}}$

$$F(x) = \mathbb{E}[f(R(x))] = \sum_{S \subseteq V} f(S) \prod_{i \in S} x_i \prod_{j \in V \setminus S} (1 - x_j),$$

2. Round to a feasible set

Overview of Techniques

Constant-Factor Violation for Non-Monotone:

- We extend the algorithm of [El Halabi et al, 2023] to non-monotone case:
  - Find a feasible solution and *randomly* divide it into two sets A and B s.t. the lower bounds are violated by at most a factor β.
  - Extend both A and B to maximize the value of the solution.
  - At least one of them is a good solution.

➔ 1/16-approx with factor-2 violation of fairness lower bounds (β=1/2)

Algorithms with Expected Fairness

- Use relax-and-round approach but *ignore fairness during rounding*
- We show that the swap rounding algorithm of [Chekuri et al., 2010] satisfies fairness bounds *in expectation*
- Is in expectation enough? All solutions can be unfair for some groups.
- What can we achieve with high probability?

$$\left(1 - \sqrt{\frac{3\ln(2C)}{\ell_c}}\right) \ell_c \leq |S \cap V_c| \leq \left(1 + \sqrt{\frac{3\ln(2C)}{u_c}}\right) u_c$$

- The violation can be large, but should be small for most applications.
- Use existing algorithms to solve the relaxed problem and its complement.
  - Approximation guarantee is optimal for monotone *f*.
  - Approximation guarantee can be as high as ¼ for non-monotone *f*.
- We show that the (1-r) factor can't be improved, even without the matroid constraint.
- Cardinality constraint: better approx than [El Halabi et al, 2020], better approx than general matroids for non-monotone *f*, no violation of fairness,

Can We Satisfy Fairness Exactly?

- **Very challenging in general:** common algorithmic techniques fail
- Relax-and-round approaches do not work: integrality gap is at least √n for special case of bipartite perfect matching
- Greedy approaches do not work: might return arbitrarily bad solution
- Swapping techniques do not work: might need to swap out all elements

Decomposable Submodular Functions

- Function *f* that can be decomposed over color or matroid groups:

$$f(S) = \sum_{G \in \mathcal{G}} f_{1,G}(S \cap G) + \sum_{c=1}^{C} f_{2,c}(S \cap V_c)$$

- **Submodular Welfare problem** is a special case
- Use relax-and-round approach again but *do not ignore fairness* during rounding
- Reduce to rounding over *intersection of two matroids*, then use corresponding swap rounding algorithm of [Chekuri et al., 2010]:
  - No violation of fairness
  - No loss in value after rounding ➡ same approximation as general case
- Hardness result for non-monotone case still applies

Open Directions

- Is there a constant-factor approximation algorithm without fairness violation for *monotone* functions?
- Is there one in the special case of maximizing over *perfect* matchings?
- **Fundamental question** [Vondrák, 2013]: approximation of maximization problem with linear objective ➡ approximation with monotone submodular objective?