Matroid Submodular Maximization

- Each element has a color encoding a sensitive attribute.
- We are given lower and upper bound constraints for each color $c$:
  \[ \ell_c \leq |S \cap V_c| \leq u_c \quad \text{for all } c = 1, \ldots, C \]

Examples: uniform matroid $|S| \leq k$, partition matroid $|S \cap V_c| \leq u_c$

Influence Maximization

Image Summarization

Fairness Setting

- Each element has a color encoding a sensitive attribute.
- We are given lower and upper bound constraints for each color $c$:
  \[ \ell_c \leq |S \cap V_c| \leq u_c \quad \text{for all } c = 1, \ldots, C \]

Find a solution such that
1. Number of blue elements in range $[1,2]$
2. Number of red elements is in range $[0,3]$
3. The solution belongs to a matroid
4. The bounds are not constants

Family of feasible sets: $F = \{ \text{feasible sets in matroid } I \}$

Prior Work

Cardinality constraint:
- Offline setting: tight $(1-1/e)$-approx for monotone $f$ [Celis et al. 2018].
- Streaming setting: 0.378-approx for non-monotone $f$, 0.192-approx for monotone $f$, and 0.192-approx for non-monotone $f$, where $\gamma = 1 - \max_e \ell_e / u_e$ [El Halabi et al., 2020].

General matroid:
- Streaming setting: 0.085-approx for monotone $f$ with factor-2 violation of fairness lower bounds [El Halabi et al., 2023].
- No results for offline setting or non-monotone objectives.

Function

<table>
<thead>
<tr>
<th>Matroid</th>
<th>Monotone</th>
<th>Approx. Ratio</th>
<th>Fairness Approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monotone</td>
<td>General</td>
<td>$1 - 1/e$ (Thm. 3.3)</td>
<td>$(1, 1)$ in expectation</td>
</tr>
<tr>
<td>Non-monotone</td>
<td>General</td>
<td>$(1 - r/e) / 4$ (El Halabi et al., 2023)</td>
<td>$(1/2, 1)$</td>
</tr>
<tr>
<td>Monotone</td>
<td>Uniform</td>
<td>$1 + \sqrt{\ln(2C)/u_e}$ (El Halabi et al., 2018a)</td>
<td>$(\beta, 1)$</td>
</tr>
<tr>
<td>Monotone</td>
<td>Uniform</td>
<td>$1 - 1/e$</td>
<td>$(1, 1)$</td>
</tr>
<tr>
<td>Monotone decomposable</td>
<td>General</td>
<td>$1 - 1/e$ (Thm. 4.4)</td>
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Lower bounds:
- No approximation better than $(1 - r)$ in sub-exponential time for non-monotone
- No $O(1/\sqrt{n})$-approximation using relax-and-round approaches

Our Results

Constant-Factor Violation for Non-Monotone:
- We extend the algorithm of [El Halabi et al, 2023] to non-monotone case:
  - Find a feasible solution and randomly divide it into two sets $A$ and $B$ s.t. the lower bounds are violated by at most $\beta$.
  - Extend both $A$ and $B$ to maximize the value of the solution.
  - At least one of them is a good solution.
- $1/16$-approx with factor-2 violation of fairness lower bounds ($\beta=1/2$)

Algorithms with Expected Fairness

- Use relax-and-round approach but ignore fairness during rounding.
- We show that the swap rounding algorithm of [Chekuri et al., 2010] satisfies fairness bounds in expectation.
- Is in expectation enough? All solutions can be unfair for some groups.
- What can we achieve with high probability?
- The violation can be large, but should be small for most applications.
- Use existing algorithms to solve the relaxed problem and its complement.
- Approximation guarantee is optimal for monotone $f$.
- Approximation guarantee can be as high as $\frac{1}{4}$ for non-monotone $f$.
- We show that the $(1-r)$ factor can't be improved, even without the matroid constraint.
- Cardinality constraint: better approx than [El Halabi et al, 2020], better approx than general matroids for non-monotone $f$, no violation of fairness.

Overview of Techniques

Can We Satisfy Fairness Exactly?

- Very challenging in general: common algorithmic techniques fail
- Relax-and-round approaches do not work: integrality gap is at least $\frac{1}{2} \ln n$ for special case of bipartite perfect matching
- Greedy approaches do not work: might return arbitrarily bad solution
- Swapping techniques do not work: might need to swap out all elements

Decomposable Submodular Functions

- Function $f$ that can be decomposed over color or matroid groups:
  \[ f(S) = \sum_{G \subseteq V} f_G(S \cap G) + \sum_{c=1}^C f_c(S \cap V_c) \]

Submodular Welfare problem is a special case
- Use relax-and-round approach again but do not ignore fairness during rounding.
- Reduce to rounding over intersection of two matroids, then use corresponding swap rounding algorithm of [Chekuri et al., 2010]:
  - No violation of fairness
  - No loss in value after rounding: same approximation as general case.
- Hardness result for non-monotone case still applies

Open Directions

- Is there a constant-factor approximation algorithm without fairness violation for monotone functions?
- Is there one in the special case of maximizing over perfect matchings?
- Fundamental question [Vondrák, 2013]: approximation of maximization problem with linear objective "\approx" approximation with monotone submodular objective.