Approximation Algorithms for the Asymmetric Traveling Salesman Problem

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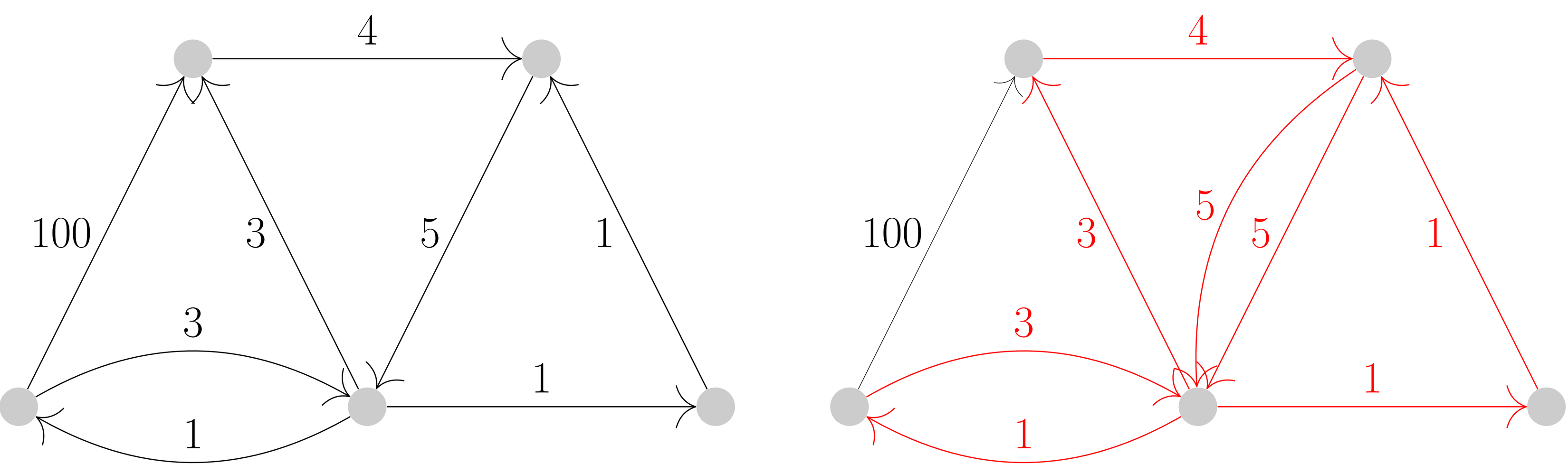
ATSP: definition

Given **directed graph** $G = (V, E)$ with costs on edges $w : E \to \mathbb{R}_+$, find **cheapest tour** which **visits all vertices**.

- Can visit vertices (even edges) multiple times.
- Best-known problem in combinatorial optimization.
- Why asymmetric costs? Consider being a postman in Lausanne...

Example



An ATSP instance and its optimum solution (in red).

Linear programming relaxation

Write x_e for the number of times we traverse edge e and

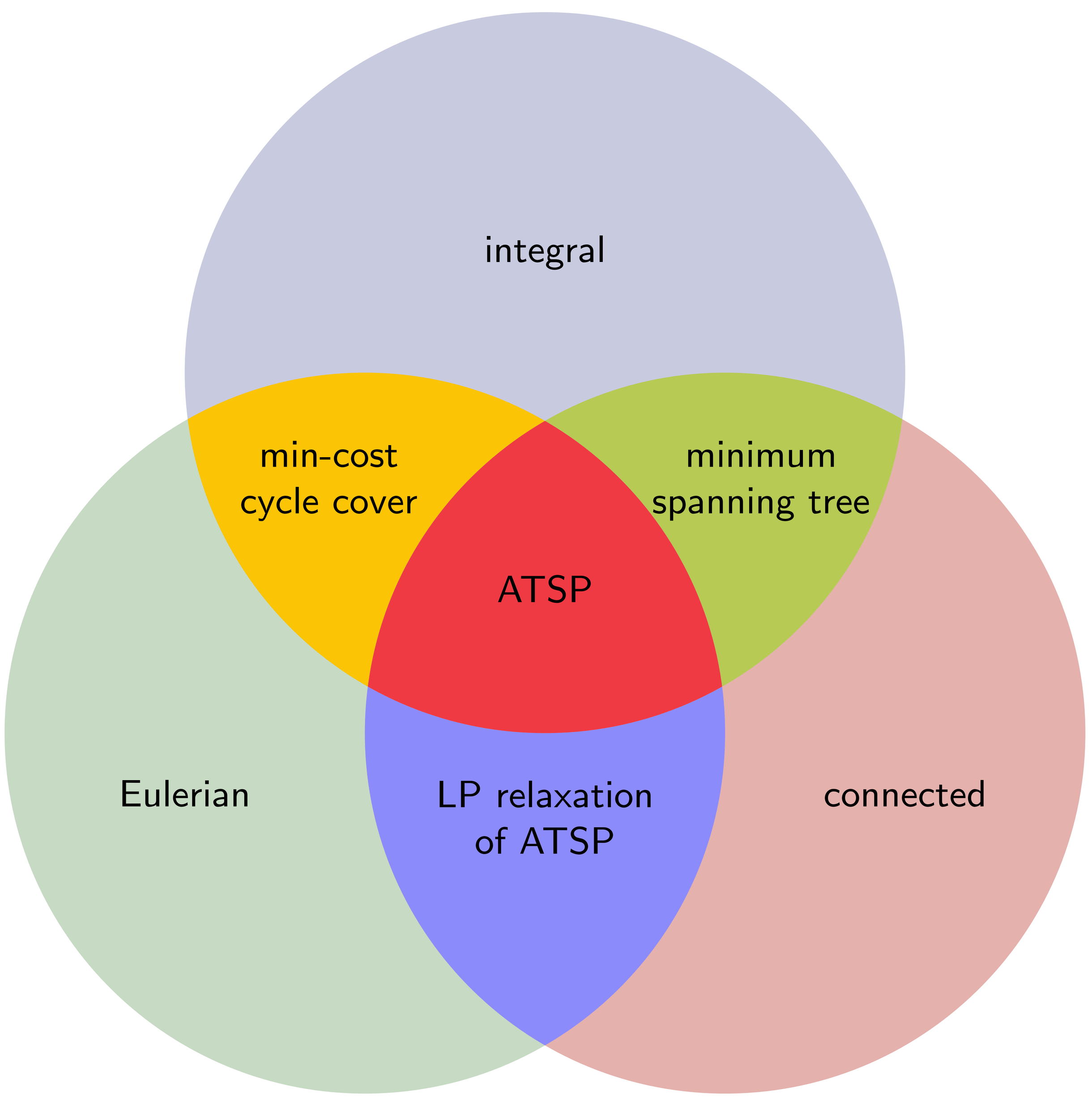
$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} w_e x_e \\ \text{subject to} \quad & x(\delta^+(v)) = x(\delta^-(v)) && \text{for all } v \in V, \\ & x(\delta^+(S)) \geq 1 && \text{for all } \emptyset \neq S \subsetneq V, \\ & x_e \geq 0 && \text{for all } e \in E. \end{aligned}$$

That is:

- x should be Eulerian,
- x should connect the entire graph.

Is this hard?

We want the cheapest $x : E \to \mathbb{R}_+$ which touches every vertex and is:



Everything in the diagram is easy, except for ATSP!

Approximation algorithms

- OPT = value of optimum solution
- Algorithm $\mathcal{A}$ is an **α-approximation** if $w(\mathcal{A}(G)) \leq \alpha \cdot OPT$ for every G
- What is the best approximation ratio possible?

Integrality gap

- OPT_{LP} = value of LP relaxation
- Clearly $OPT_{LP} \leq OPT$, but how good is this lower bound?
- **Integrality gap** = max ratio OPT/OPT_{LP}: quality of the lower bound
- Why bother?
  - Design of LP rounding algorithms
  - Integrality gap $\leq \alpha \implies$ exists α-estimation algorithm
  - Integrality gap $> \alpha \implies$ exists no α-approximation with respect to this relaxation

Known results

- Approximation algorithms:
  - $\mathcal{O}(\log n / \log \log n)$-approximation algorithm [AGM+ '10]
  - lower bound: 75/74-approximation is NP-hard [KLS '13]
- Integrality gap:
  - upper bound: $\mathcal{O}(\text{poly} \log \log n)$ [AG '14]
  - lower bound: 2 [CGK '06]
- Is there an $\mathcal{O}(1)$-approximation algorithm?

Local-Connectivity ATSP

- For **unweighted** graphs (all costs = 1): **yes**! [Sve '15] showed:
- New, easier problem called Local-Connectivity ATSP
- **Reduction:** if can approximate Local-Connectivity ATSP well, then can also approximate ATSP well
- Can indeed approximate Local-Connectivity ATSP well for unweighted graphs
- [STV '16]: can also do it for **graphs with two edge costs**

Future work

- For what wider classes of graphs is there an $\mathcal{O}(1)$-approximation? Even the case of three edge costs is unsolved.
- What about the general case? Bridge the gaps!
- Can we get an algorithm which matches the known integrality gap upper bound?
- In the symmetric version: can we beat 1.5-approximation (a result from the 70's)?

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